Lecture 11

- Need a more robust definition of Fourier transform to deal with common signals, ones for which classical definition will not do. Two issues:
  - Convergence of integral: define the F.T.
  - Fourier inversion
- Two ways of dealing with these problems.
  - Ad hoc, special techniques.
  - Rework the foundations and definition.
- Problem is evident in the very first example, $f(t) = \pi(t)$. F.T. is fine

$$F\pi(s) = sinc(s) = \frac{\sin(\pi s)}{\pi s}$$

  - Problem is $F^{-1} sinc = \pi$. Or equivalently, by duality, $F sinc = \pi$.

$$F^{-1} sinc(t) = \int e^{2\pi i s t} sinc(s) ds = \begin{cases} 1, & |t| \leq 1/2, \\ 0, & |t| > 1/2. \end{cases}$$

  - The integral result is OK but requires special techniques. (In fact, special problem of end points $s = \pm\frac{1}{2}$)
- 2^{nd} example, $f(t) = 1$. No way to make sense of

$$\int_{-\infty}^{\infty} e^{-2\pi i s t} 1 dt$$

  Likewise, $\sin 2\pi t, \cos 2\pi t$ can't make sense.
- Analoguous to Fourier series, we need a new conception of convergence.
- How to choose basic phenomena which can be used to explain the others?
  - Back away from problems. What is the best situation? — Identify the "best signals" for Fourier transform.
  - Call the class of signals S. We want:
    - 1) If $f(t)$ is in S, then Ff is defined and Ff is also in S. (For Fourier transform works.)
    - 2) Fourier inversion works. $F^{-1}Ff = f$ and $FF^{-1}f = f$.
    - Further property: Parsexal's identity

$$\int_{-\infty}^{\infty} |g(s)|^2 ds = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

  - How to define S? — Solved by Laurant Schwartz. S is class of **rapidly decreasing functions**.
    - 1) $f(x)$ is infinitely differentiable.

- 2) For any $m, n \geq 0$, $|x|^n|\frac{d^m}{dx^m}f(x)| \to 0$ as $x \to \pm\infty$. (Characterize the rapidly decreasing functions. Any derivative tends to zero faster than any power of x).
- Any such functions? Gaussian is such one,

$$f(x) = e^{-\pi x^2}$$

C infinitely differentiable functions which are 0 outside a finite interval, "compact support" (bounded and closed).
- Connection comes via derivative theorem

$$F(f^{(m)})(s) = (2\pi is)^m Ff(s)$$

If f goes to 0 and power of s goes to infinity, and the whole RHS goes to zero, then it characterise how the LHS changes, which is how the derivative changes.
- By Parsexal's identity, both integral converges because of the rapid decrease.
- The following equation holds.

$$\int_{-\infty}^{\infty} Ff(s)\overline{Fg(s)}ds = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$$

$$\begin{aligned}\int_{-\infty}^{\infty} Ff(s)\overline{Fg(s)}ds &= \int_{-\infty}^{\infty} Ff(s)(\int_{-\infty}^{\infty} e^{2\pi ist}\overline{g(t)}dt)ds \\ &= \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} Ff(s)e^{2\pi ist}ds)\overline{g(t)}dt \\ &= \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt\end{aligned}$$

Lecture 12

- S is so good due to the two properties. *But π, trig, constant are not in this class*.
- How can we be sure we have it lost anything, and will gain greater generality? To answer this, pick up another line of development, i.e., the idea of "generalized functions", (also known as "**distributions**"), typified by Dirac δ.
  - Define this way

$$\begin{aligned}(a) \quad & \delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & otherwise \end{cases} \\ (b) \quad & \int_{-\infty}^{\infty} \delta(x)dx = 1 \\ (c) \quad & \int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0)\end{aligned}$$

- δ is supposed to represent a function which is concentrated at a point. Various ways of doing this, but it's always via a limiting process.
  - E.g. Consider family of rectangle functions. Think of $\frac{1}{\epsilon}\pi_\epsilon(x)$. The area is one (considering the thing operationally).

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\epsilon}\pi_\epsilon(x)\phi(x)dx &= \int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon}\phi(x)dx \\
&= \frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2} (\phi(0) + \phi'(0)x + \phi''(0)x^2 + \dots)dx \\
&= \phi(0) + O(\epsilon) \\
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\pi_\epsilon(x) \text{ does not make sense}
\end{aligned}
$$

$$
\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \frac{1}{\epsilon}\pi_\epsilon(x)\phi(x)dx = \phi(0)
$$

which makes sense, operationally.

- To capture this idea, and to include more, we need to change a point of view.
- Focus is an outcome rather than the process.

- **Definition of distributions**
  1. Start with class of "test functions" ϕ ("best functions" for given area).
  2. Associated with test functions is class of distributions.

     A distribution T is a linear functional on test functions. For a test function ϕ, $T(\phi)$ is a number and T is linear in that

     $$
     \begin{aligned}
     T(\phi_1 + \phi_2) &= T(\phi_1) + T(\phi_2) \\
     T(\alpha\phi) &= \alpha T(\phi)
     \end{aligned}
     $$

  3. Continuity $\phi_n \to \phi \Rightarrow T(\phi_n) \to T(\phi)$.
- Often say that a distribution is paired with a test function. Often write

  $$
  < T, \phi > \text{ or } T(\phi)
  $$

- Rediscover δ in this context of distribution
  - Operationally, the effect of δ is to evaluate a function at the origin.
  - Define δ by $< \delta, \phi >= \phi(0)$

    $$
    \begin{aligned}
    < \delta, \phi_1 + \phi_2 > &= (\phi_1 + \phi_2)(0) \\
    &= \phi_1(0) + \phi_2(0) \\
    &= < \delta, \phi_1 > + < \delta, \phi_2 >
    \end{aligned}
    $$

    $$
    \begin{aligned}
    \phi_n \to \phi &\Rightarrow < \delta, \phi_n > \to < \delta, \phi > \\
    \phi_n \to \phi &\Rightarrow \phi_n(0) \to \phi(0)
    \end{aligned}
    $$

- Define **shift** δ: δ_a as a distribution by $< \delta_a, \phi >= \phi(a)$.
- How to consider "ordinary functions" in this context? E.g., How to consider the constant function 1 as a distribution? Given a test function ϕ, how do we define a pairing of 1 and ϕ?
  - Pairing is by integration

    $$
    < 1, \phi >= \int_{-\infty}^{\infty} 1\phi(x)dx
    $$

  - Likewise, for example,

$$< \pi, \phi >= \int_{-\infty}^{\infty} \pi(x)\phi(x)dx$$

- You can't integral $\sin 2\pi x$ from $-\infty$ to ∞, but if you multiply it with a decreasing or dying function to make them as a whole converge, then the integral holds

$$< \sin 2\pi x, \phi(x) >= \int_{\infty}^{\infty} \sin 2\pi x \phi(x)dx$$

- $f(x)$ is "any" function. Consider $f(x)$ as a generalized function by defining (a linear operation)

$$< f, \phi >= \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

  If you provide a very good test function ϕ, then your function can be very wild. In this case, you can still get a converged function.

- To see how a function defines its distribution you can see how it operates on the test function through the integral.

Lecture 13

- *Fourier Transform as a distribution set-up*
  - First define class of **test functions**: typically have particularly nice properties
  - For the Fourier transform, take rapidly decreasing functions
  - A distribution (generalized function) is a continuous linear functional on test functions
  - ϕ is a test function ad T is a distribution. Write $< T, \phi >$. T operates on ϕ, like T is a measuring device and ϕ is something you are measuring.
  - T is linear:

$$< T, \phi_1 + \phi_2 >=< T, \phi_1 > + < T, \phi_2 >$$
$$< T, \alpha\phi >= \alpha < T, \phi >$$

  - Continuity:

$$\phi_n \to \phi \Rightarrow < T, \phi_n > \to < T, \phi >$$

  - The hard mathematical work is to define $\phi_n \to \phi$. The more properties you declare on a function, the harder it can satisfy such convergence.
  - The distributions is the *dual space* of the space of test functions.
- **2nd example**: Distribution induced by functions. If $f(x)$ a functoin such that $\int_{-\infty}^{\infty} f(x)\phi(x)dx$ makes sense D, then ϕ is such a good function that can be a test function. Define a pairing of f and ϕ

$$< f, \phi >= \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

  - Many functions induces distributions. For example,

$$\int_{-\infty}^{\infty} \phi(x)dx = \int_{-\infty}^{\infty} 1 \cdot \phi(x)dx < \infty$$
$$< 1, \phi >= \int_{-\infty}^{\infty} \phi(x)dx$$
$$\int_{-\infty}^{\infty} e^{2\pi iax}\phi(x)dx < \infty$$
$$< e^{2\pi iax}, \phi >= \int_{-\infty}^{\infty} e^{2\pi iax}\phi(x)dx$$

- Take test functions to be S rapidly decreasing functions.
$$\phi \in S \Rightarrow F\phi \in S \text{ and } F^{-1}\phi \in S$$
$$FF^{-1}\phi = \phi$$
$$F^{-1}F\phi = \phi$$
    - Corresponding class of distributions are called **tempered distribution**.
    - T a tempered distribution, want to define FT a tempered distribution.
    - Have to define $< FT, \phi >$. HOW?
    - What is pairing by integration?
$$\begin{aligned} < FT, \phi > &= \int_{-\infty}^{\infty} FT(x)\phi(x)dx \\ &= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} e^{-2\pi iyx}T(y)dy)\phi(x)dx \\ &= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} e^{-2\pi iyx}\phi(x)dx)T(y)dy \\ &= \int_{-\infty}^{\infty} F\phi(y)T(y)dy \\ &=< T, F\phi > \end{aligned}$$
    - The RHS makes sense. ϕ is a Schwartz function and $F\phi$ is as well. Then T operates on a Schwartz function. That is fine. Turn this into a definition.
    - T a tempered distribution. Define FT by
$$< FT, \phi >=< T, F\phi >$$
    - Inverse Fourier Transform
$$< F^{-1}T, \phi >=< T, F^{-1}\phi >$$
    - Prove the theorem
$$< F^{-1}FT, \phi >=< FT, F^{-1}\phi >=< T, FF^{-1}\phi >$$
$$< FF^{-1}T, \phi >=< F^{-1}T, F\phi >=< T, F^{-1}F\phi >$$
- Calculate the $F\delta$:

$$
\begin{aligned}
< F\delta, \phi > &=< \delta, F\phi > \\
&= F\phi(0) \\
&= \int_{-\infty}^{\infty} e^{-2\pi i 0 x}\phi(x)dx \\
&=< 1, \phi > \\
\Rightarrow F\delta = 1
\end{aligned}
$$

- δ "infinitely concentrated". Its Fourier transform is uniformly spread out by $F\delta = 1$. It is a case of spread theorem in that concentration in time domain indicates uniform spreadout in frequency domain.
- Calculate $F\delta_a$:

$$
\begin{aligned}
< F\delta_a, \phi > &=< \delta_a, F\phi > \\
&= F\phi(a) \\
&= \int_{-\infty}^{\infty} e^{-2\pi i a x}\phi(x)dx \\
&=< e^{-2\pi i a x}, \phi > \\
\Rightarrow F\delta_a = e^{-2\pi i a x}
\end{aligned}
$$

- Calculate $Fe^{2\pi i a x}$:

$$
\begin{aligned}
< Fe^{2\pi i a x}, \phi > &=< e^{2\pi i a x}, F\phi > \\
&= \int_{-\infty}^{\infty} e^{2\pi i a x} F\phi(x)dx \\
&= \phi(a) \\
&=< \delta_a, \phi > \\
\Rightarrow Fe^{2\pi i a x} = \delta_a
\end{aligned}
$$

- How about $\sin$ and $\cos$?

$$
\begin{aligned}
\cos(2\pi a x) &= \frac{e^{2\pi i a x} + e^{-2\pi i a x}}{2} \\
F\cos(2\pi a x) &= \frac{\delta_a + \delta_{-a}}{2} \\
\sin(2\pi a x) &= \frac{e^{2\pi i a x} - e^{-2\pi i a x}}{2i} \\
F\cos(2\pi a x) &= \frac{\delta_a - \delta_{-a}}{2i}
\end{aligned}
$$

Lecture 14

- **Derivative of a distribution**
  - T is a given distribution, how to define T'.
  - Have to define $< T', \phi >$ and ϕ is a test function.
  - If T' were given by a function, we will have

$$
\begin{aligned}
< T', \phi > &= \int_{-\infty}^{\infty} T'(x)\phi(x)dx \\
&= T(x)\phi(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T(x)\phi'(x)dx \\
&= -\int_{-\infty}^{\infty} T(x)\phi'(x)dx \\
&= - < T(x), \phi'(x) >
\end{aligned}
$$

- Turn this into a definition: Define T' by
$$< T', \phi > = - < T, \phi' >$$

- Example
  - $u(x) = 1, x > 0; 0$ otherwise. $u(x)$ defines a distribution since
$$< u, \phi > = \int_{-\infty}^{\infty} u(x)\phi(x)dx = \int_{0}^{\infty} \phi(x)dx$$

    The integral exists.
  - So u' exists as a distribution.
$$
\begin{gathered}
< u', \phi > = - < u, \phi' > \\
= -\int_{-\infty}^{\infty} u(x)\phi'(x)dx = -\int_{0}^{\infty} \phi'(x)dx \\
= \phi(0) = < \delta, \phi >
\end{gathered}
$$

  - $sgn(x)$:
$$
\begin{gathered}
< sgn', \phi > = - < sgn, \phi' > \\
= -\int_{-\infty}^{\infty} sgn(x)\phi'(x)dx \\
= -(-\int_{-\infty}^{0} \phi'(x)dx + \int_{0}^{\infty} \phi'(x)dx) \\
= 2\phi(0) = < 2\delta, \phi >
\end{gathered}
$$

  - **Applications to Fourier transform - Derivative Theorem**
$$
\begin{gathered}
F(T') = 2\pi isFT \\
(FT)' = F(-2\pi itT)
\end{gathered}
$$

  - Example
$$
\begin{gathered}
sgn' = 2\delta \\
F(sgn') = 2\pi isF(sgn) = 2 \\
F(sgn) = \frac{1}{\pi is}
\end{gathered}
$$

  - Need more argument — see the notes
- **Multiplication and Convolution**: multiplication of functions *does not* carry over to multiplication of distributions.

    - S and T are distributions. The ST is generally not defined.
    - What is defined is fT where f is a function, not a distribution.
    - How to define $< fT, \phi >$.
    - If T given by a function, then

$$
\begin{aligned}
< fT, \phi > &= \int_{-\infty}^{\infty} f(x)T(x)\phi(x)dx \\
&= \int_{-\infty}^{\infty} T(x)f(x)\phi(x)dx \\
&= < T, f\phi >
\end{aligned}
$$

    - So in general, define fT by $< fT, \phi >=< T, f\phi >$.
    - This makes sense only if $f\phi$ is a test function.
    - We used this when we wrote $F(T') = 2\pi isFT$.
    - *Special cases*

$$
\begin{aligned}
&< f\delta, \phi >=< \delta, f\phi > \\
&= f(0)\phi(0) =< f(0)\delta, \phi > \\
&\Rightarrow f\delta = f(0)\delta
\end{aligned}
$$

      More generally, $f\delta_a = f(a)\delta_a$.

      This is the **sampling property** of δ.

- **Convolution**
    - S, T distributions, how to define $S * T$? — Not always defined.
    - Need extra restrictions on S and T.
    - Can define $S * T$ via pairing, but need extra conditions.
    - Many cases when all is well, e.g. $f * T$ often makes sense when f is an arbitrary function.
    - And convolution theorem holds

$$
F(f * T) = (Ff)(FT)
$$

    - **Special case** when $T = \delta$:

$$
f * \delta = f
$$

    - More generally, $(f * \delta_a)(x) = f(x - a)$.
    - Can convolve δ with itself

$$
\begin{gathered}
\delta_a * \delta_b = \delta_{a+b} \\
(f * \delta_a) * \delta_b = f(x-a) * \delta_b = f(x-a-b)
\end{gathered}
$$

    - **Scaling property** of δ
        - What is $\delta(ax)$

$$
\begin{aligned}
< \delta(ax), \phi(x) > &= \int_{-\infty}^{\infty} \delta(u)\phi(u/a)/adu \\
&= \frac{1}{a}\int_{-\infty}^{\infty} \delta(u)\phi(u/a)du \\
&= \frac{1}{a} < \delta, \phi(\frac{u}{a}) > \\
&= \frac{1}{a}\phi(0) \\
&= \frac{1}{a} < \delta, \phi >
\end{aligned}
$$

- Similar argument if $a < 0$, get $\delta(ax) = \frac{1}{|a|}\delta(x)$.

Lecture 15 Fourier transform and diffraction

- Diffraction = interference patterns of light through holes.
- It involves
  - light from a distant source.
  - plane with apertures.
  - At some distrance, you have an image plane.
  - See diffraction patterns.
  - We'll assume light is an oscillating EM field.
  - monochromatic
  - Distance of image plane determines
    - near field (Fressel diff)
    - far field (Frounhoffer diff)
    - measure distance relative to wavelength
- Distant source means that the aperture plane is a wavefront.
- So wave has the same phase of all pts of aperture plane.
- Represent light on aperture plane as $Ee^{2\pi ivt}$. E is strength of the field. v is the frequency.
- Assume E is constant, say E_0, an aperture plane.
- What is the electric field at a point P on the image plane?
  - Lights i.e., the wave gets to P along different path. How does the results add up?
  - Approach this via Hoyghens' Principle.
  - Each point on a wave front can be regarded as a new source.
  - Add up (integrate) effects of all sources.
  - Aperture width dx. Strength of field at x is $E_0e^{2\pi ivt}dx$.
  - Main change in going from x to P is *change in phase*.
  - Phase change in traveling a distance r from x to P.

- In a distance r, wave goes through $\frac{r}{\lambda}$ cycles, λ is wavelength.
- So phase change is $\frac{2\pi r}{\lambda}$ and so field at P due to field at x is

$$dE = E_0 e^{2\pi i v t} e^{-2\pi i r/\lambda} dx$$

- Total field: $\int_{aperture} E_0 e^{2\pi i v t} e^{-2\pi i r/\lambda} dx$
- You measure $|E|$. Quantity of interest is $\int_{aperture} d^{-2\pi i r/\lambda} dx$. NOT USEFUL.

- Now bring in Fraunhoffer approximation. Assume $r >> x$. Then

$$r_0 - x \sin\theta \approx r$$

- Plug this into the integral giving the field at P.

$$\int e^{-2\pi i \frac{1}{\lambda}(r_0 - x\sin\theta)} dx$$
$$= \int e^{-2\pi i \frac{r_0}{\lambda}} e^{2\pi i \frac{x \sin\theta}{\lambda}} dx$$
$$= \int_{-\infty}^{\infty} e^{2\pi i x P} A(x) dx$$

- This is $F^{-1} A(p)$, where $A(x)$ is the aperture function.
- For far field diffraction, the intensities of the light is the magnitude of the Fourier transform of aperture function.

Lecture 16 Crystal Grazing

- Diffraction patterns are determined by Fourier transform of the aperture function.
- Set-up
  - X ray were discoverd in 1895 by Roeatgen.
  - What are they? Are they waves?
  - If so, the wavelength should be around 10^{-8} cm. Too small to measure.
  - Crystals. What are they?
  - Macro structure determined by atomic structure.
  - Atoms arranged on a lattice. How to test this?
  - Max von lan in 1912: study atomic structure of crystals via diffraction experiments with X-rays.
- Hypothesis
  - X rays are waves so will diffract.
  - Crystals have lattice atomic structure.
  - Spacing of atomic camparable to wavelength of X rays.
- 1-Dim picture
  - atoms lay a line, effectively an infinite array
  - Periodized version of electronic density for a single atom.
  - Single density $\rho(x)$,

$$\rho_p(x) = \sum_{k=-\infty}^{\infty} \rho(x - kp)$$

- Write $\rho_p(x)$ as a convolution

$$\rho(x - kp) = p(x) * \delta(x - kp)$$
$$\rho_p(x) = \sum_{k=-\infty}^{\infty} p(x) * \delta(x - kp) = p(x) * \sum_{k=-\infty}^{\infty} \delta(x - kp)$$

- Use notation

$$\text{III}_p(x) = \sum_{-\infty}^{\infty} \delta(x - kp)$$
$$\rho_p(x) = p(x) * \text{III}_p(x)$$
$$F\rho_p = (F\rho)(F\text{III}_p)$$

- What is $F\text{III}_p$?
- Take p=1. III makes sense as a distribution.

$$< \text{III}, \phi >= \sum_{k=-\infty}^{\infty} \phi(k)$$

Sum converges. So all is well. So $F\text{III}$ makes sense.

$$< F\text{III}, \phi >=< \text{III}, F\phi >= \sum_{k=-\infty}^{\infty} F\phi(k)$$

- Could write down

$$F\text{III} = \sum_{k=-\infty}^{\infty} F(\delta(x - k)) = \sum_{k=-\infty}^{\infty} e^{-2\pi iks}$$

It doesn't converge classically, but it is ok to be a distribution. It misses the point, the essence of integers.

- Need **Poisson summation formula**. ϕ rapidly function, then

$$\sum_{k=-\infty}^{\infty} \phi(k) = \sum_{k=-\infty}^{\infty} F\phi(k)$$

Proof. Periodise ϕ to have period 1.

$$\Phi(x) = \sum_{k=-\infty}^{\infty} \phi(x - k)$$

Expand Φ in Fourier series

$$\Phi(x) = \sum_{k=-\infty}^{\infty} \hat{\Phi}(k) e^{2\pi ikx}$$

Know $\hat{\Phi}(k) = F\Phi(k)$, so

$$\Phi(x)=\sum_{k=-\infty}^{\infty}\phi(x-k)=\sum_{k=-\infty}^{\infty}F\Phi(k)e^{2\pi ikx}$$

Evaluate at $x=0$

$$\Phi(0)=\sum_{k=-\infty}^{\infty}F\phi(k)=\sum_{k=-\infty}^{\infty}\phi(-k)=\sum_{k=-\infty}^{\infty}\phi(k)$$

- Back to $F\text{III}$,

$$\begin{aligned}<F\text{III},\phi>=<\text{III},F\phi>\\ =\sum_{k=-\infty}^{\infty}F\phi(k)\\ =\sum_{k=-\infty}^{\infty}\phi(k)\\ =<\text{III},\phi>.\\ F\text{III}=\text{III}\end{aligned}$$

- Let's do $F\text{III}_p$

$$\begin{aligned}\text{III}_p(x)=\sum_{k=-\infty}^{\infty}\delta(x-kp)\\ =\frac{1}{p}\text{III}\left(\frac{x}{p}\right)\\ F\,\text{III}_p\,(x)=\frac{1}{p}F(\text{III}(\frac{x}{p}))\\ =\text{III}(px)\\ \text{III}\,(px)=\sum_{k=-\infty}^{\infty}\delta(px-k)=\sum_{k=-\infty}^{\infty}\delta(p(x-k/p)\\ =\frac{1}{p}\sum_{k=-\infty}^{\infty}\delta(x-k/p)\\ F\,\text{III}_p=\frac{1}{p}\text{III}_{1/p}\end{aligned}$$

- Back to crystal

$$\begin{aligned}F\rho_p=(F\rho)(F\,\text{III}_p)\\ =\frac{1}{p}\sum_{k=-\infty}^{\infty}F\rho(\frac{k}{p})\delta(x-\frac{k}{p})\end{aligned}$$

So the spacing in crystal is p due to the reciprocal property.

Lecture 17 Sampling and Interpolation

- $\text{III}_p(x)=\sum_{k=-\infty}^{\infty}\delta(x-kp)$.
- Three important properties
  - sampling properties

$$f(x)\,Ш_p(x)=\sum_{k=-\infty}^{\infty} f(kp)\delta(x-kp)$$

- Perioding property

$$(f * Ш_p)(x)=\sum_{k=-\infty}^{\infty} f(x-kp)$$

- Fourier transform property

$$F\,Ш_p=\frac{1}{p}\,Ш_{1/p}$$
$$F^{-1}\,Ш_p=\frac{1}{p}Ш_{1/p}$$

- Set up the "interpolation problems"

- Will be able to interpolate all values of a signal or function from a discrete set of samples.
  - Set-up: Imagine have a process evolving in time. You make a set of measurements at equal time intervals.

$$(t_0,y_0),(t_1,y_1),(t_2,y_2),\ldots$$

  - You wish ask:
    - Fit a curve to the data.
    - Interpolate values of process (fun) at intermediate points based on the measurements.
    - Many possibilities.
  - If possible, might make more measurements to suggest better fit to make more accurate interpolation.
  - *More rapid bends, the more uncertainty you have in your curve fitting or interpolation.*
  - We want to regulate how rapidly the function is "oscillating" between sample values.
  - Governed by Fourier transform.
  - One possibility is to eliminate all frequencies beyond a certain point — by assumption.
  - Make this a definition.
- **A function $f(t)$ is bandlimited if $Ff(s)=0$ for $|s|\geq p/2$ for some p. The smallest such p is called the bandwidth.**
- **For bandlimited signals, can solve interpolation problem *exactly*!!!**
  - Get a formula for $f(t)$, for all t in terms of values $f(t_k)$ at discrete set of points t_k.
  - Suppose Ff is bandlimited with bandwidth p.
  - $Ff=\pi_p(Ff * Ш_p)$

$$f=F^{-1}(Ff)$$
$$=F^{-1}\pi_p F^{-1}(Ff * Ш_p)$$
$$=$$

  -

Lecture 18

- Points k/p sample points, spaced $1/p$ apart.
- Shifted sinc's $sinc(p(t-k/p))$: interpolating functions.
- Call p the sampling rate. Speak in terms of "samples per second". Also call p the "Nyquist rate".
- Can sample at the rate higher than p. All is well with the rest of derivation. Samples points spaced closer together.
- What happens if sample at too low rate? Later--
- $f(t) = \sum_{k=-\infty}^{\infty} f(k/p) sinc(p(t-k/p))$
- Interpolation involves infinite number of sample points.
- *Need to use a finite sum in any real application. Introduces error.*
- Deeper phenomenon
  - A signal cannot be both limited in time and in frequency.
  - If $Ff(s) = 0$ for $|s| \geq p/2$ then $f(t) \neq 0$ for t large.
  - Likewise, if $f(t) = 0$ for $|t| \geq q/2$ then $Ff(s) \neq 0$ for s large.

$$< \textit{Empty Math Block} >$$

- Real world signals are limited in time and in frequency.

Aliasing and Interpolation

- $f(t)$ is bandlimited.
- What happens if you try this for a p' that's too small.
- You can form $Ff * \text{Ш}_{p'}$, shifted and sum up.
- Can cut off by $\pi_{p'}$. You don't get back Ff.
- You persevere. Take inverse Fourier transform. Get a sampling formula.
- Cannot distinguish two functions, aliasing happens.
- Finally, $Ff \neq \pi_{p'}(Ff * \text{Ш}_{p'})$.
- Example
  - $f(t) = \cos(\frac{9\pi}{2}t)$ with frequency $9/4$. But we can only sample with $p = 1$.
  - $g(t) = \cos(\pi t/2)$.
  - Find that $g(t) = f(t)$ at $t = 0, \pm 1, \pm 2, \ldots$

Lecture 19 Continuous to Discrete: DFT

- Sampling music: can hear up to approximately 20,000 Hz.
  - $p = 40,000$ sampling rate for music.
  - In fact, sampling rate for CDs is 44.1 kHz.

    - 16 kHz introduces aliasing.
- Welcome to the Modern World.
- Want to introduce the discerete Fourier transform (DFT).
- Moving from continuous and analog to discrete and digital.
- Plan
    - Find a reasonable discrete approximation to $f(t)$.
    - Find a reasonable discrete approximation of $Ff(s)$.
    - Find a reasonable way passing from the discrete form of f to discrete form of $Ff(s)$.
- Base this on misuse of sampling.
- Assume
    - $f(t)$ is limited to $0 \leq t \leq L$.
    - $Ff(s)$ is limited to $0 \leq s \leq 2B$.
    - This cannot be. ($Ff(s)$ limited to $0 \leq s \leq 2B$)
    - Saying this to make *indexing* of discrete variable easier.
- To get a reasonable discrete approximation of f by sampling, take samples spaced $\frac{1}{2B}$.
    - $N = (2B)L$
    - $t_0 = 0, t_1 = \frac{1}{2B}, t_2 = \frac{2}{2B}, \ldots, t_{N-1} = \frac{N-1}{2B}$.
    - Sampled form of $f(t)$

$$f(t) \sum_{k=0}^{N-1} \delta(t - t_k) = \sum_{k=0}^{N-1} f(t_k) \delta(t - t_k)$$

      called $f_{sampled}(t)$.
    - Take Fourier transform of this $Ff_{sampled}(s) = \sum_{k=0}^{N-1} f(t_k) e^{-2\pi i s t_k}$.
    - Want to sample $Ff_{sampled}$.
    - How to sample $Ff_{sampled}$ in the frequency domain so get reasonable discrete version?
    - Take samples spaced $1/L$.
    - Again, M points in freq domain. $M = 2BL = N$, the number of sample points in time domain.
    - In frequency domain, again take N sample points spaced $1/L$ apart.

$$s_0 = 0, s_1 = 1/L, s_2 = 2/L, \ldots, s_{N-1} = \frac{N-1}{L}.$$

    - Sampled form of $Ff_{sampled}$

$$
\begin{aligned}
&(Ff_{sampled}(s))\sum_{m=0}^{N-1}\delta(s-s_m) \\
&= \left(\sum_{k=0}^{N-1} f(t_k)e^{-2\pi i s t_k}\right)\left(\sum_{m=0}^{N-1}\delta(s-s_m)\right) \\
&= \sum_{k,m=0}^{N-1} f(t_k)e^{-2\pi i s_m t_k}\delta(s-s_m)
\end{aligned}
$$

- Sampled values of $Ff_{sampled}$ are

$$
\begin{aligned}
F(s_0) &= \sum_{k=0}^{N-1} f(t_k)e^{-2\pi i s_0 t_k} \\
F(s_1) &= \sum_{k=0}^{N-1} f(t_k)e^{-2\pi i s_1 t_k} \\
&\cdots \\
F(s_{N-1}) &= \sum_{k=0}^{N-1} f(t_k)e^{-2\pi i s_{N-1} t_k}
\end{aligned}
$$

  This is discrete approximation to Fourier transform of discrete approximation of $f(t)$.
- $f(t)$ is discretized to $f(t_0),\ldots,f(t_{N-1})$. $Ff(s)$ is discretized to $F(s_0),\ldots,F(s_{N-1})$ where $F(s_m)=\sum_{k=0}^{N-1} f(t_k)e^{-2\pi i s_m t_k}$.
- Still see the continuous picture.
- Final step in defining DFT is to eliminate the continuous and use only discrete signals.
- In our set-up, $t_k=\frac{k}{2B}, s_m=\frac{m}{L}, t_k s_m=\frac{km}{2BL}=\frac{km}{N}$. Then the complex exponential $e^{-2\pi i s_m t_k}=e^{-2\pi i km/N}$.
- Just see the indices k and m.
- Finally identify $f(t_k)$ with value f[k] of discrete signal. f = (f[0],...,f[N-1]), f[k]=$f(t_k)$.
- Likewise, replace s_m by index m, i.e. F discrete signal. F=(F[0],...,F[N-1]), F[m]=$Ff(s_m)$.
- *If we do this, then all traces of continuous variables are gone and have transformation from one discrete signal to another discrete signal.*
- That is to say, if we start off the f= (f[0],...,f[N-1]). DFT of f is the discrete signal of F = (F[0],...,F[N-1]), where F[m]= $\sum_{k=0}^{N-1} f[k]e^{-2\pi i mk/N}$.

Lecture 20

- Definition of DFT
- Three quantities of interest, $\Delta t, \Delta s, N$ such that

$$
\begin{aligned}
N\Delta t &= L \\
N\Delta s &= 2B \\
\Delta t\Delta s &= \frac{L}{N}\frac{2B}{N}=\frac{2BL}{N^2}=\frac{1}{N}
\end{aligned}
$$

  "Reciprocity relationship"

- You can imagine choosing
  - Δt - how frequent you sample.
  - N - the number of samples.
  - The Δs is determined. So "resolution in freq" is fixed.
- Back to the discrete setting.
- Want to make the DFT and associated formulas look like continuous case.
- Want to realize the complex exp as arising from a discretet signal. The discrete (vector) complex exponential.
- Represent

$$w = [1, e^{2\pi i/N}, e^{2\pi i2/N}, \dots, e^{2\pi i(N-1)/N}]$$

  then

$$Ff[m] = \sum_{n=0}^{N-1} f[n] w^{-n}[m]$$
$$Ff = \sum_{n=0}^{N-1} f[n] w^{-n}$$

- The important property of discrete complex exponential: orthogonality

$$w^k \cdot w^l = \begin{cases} 0, & k \neq l, \\ N, & k = l. \end{cases}$$

- The consequence of orthogonality is inverse DFT.

$$F^{-1}f = \frac{1}{N} \sum_{n=0}^{N-1} f[n] w^n$$
$$F^{-1}f[m] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] w^n[m]$$