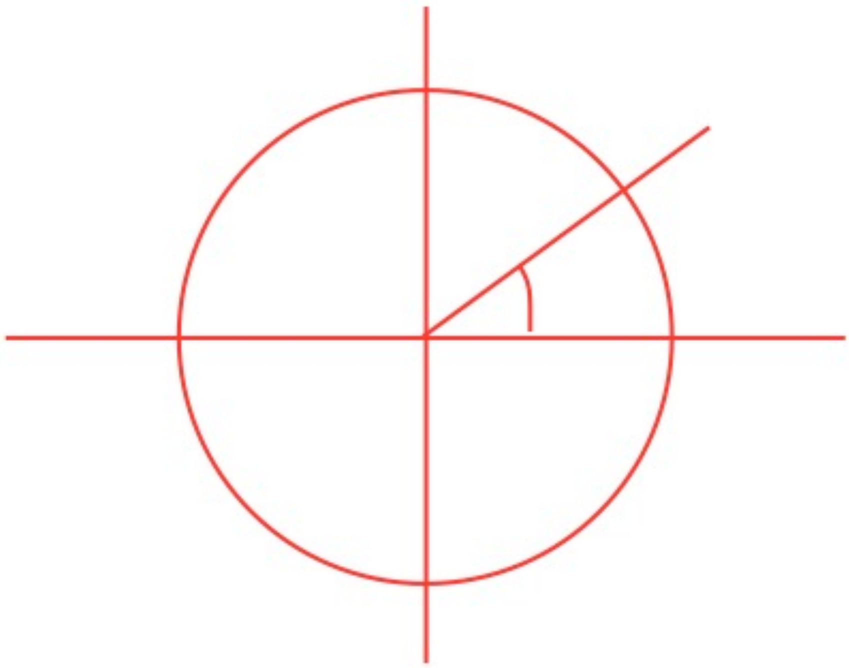
Lecture 1 Fourier Series

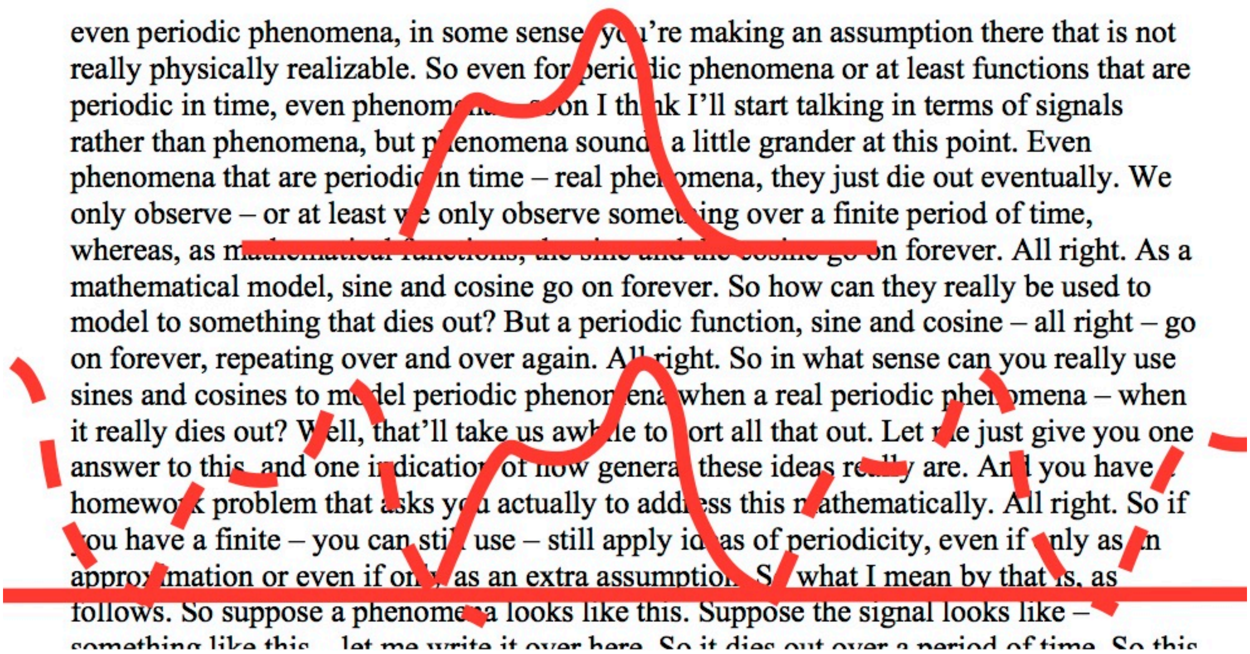
- *Fourier series* is identified with mathematical analysis of periodic phenomena.
- *Fourier transform* as a limiting case of Fourier series is concerned with non-periodic phenomena.
- Some ideas carry back and forth, some don't.
- Signals and functions essentially mean the same thing.
- *Analysis and synthesis*
  - analysis: break up a signal into simpler constituent parts.
  - synthesis: reassemble a signal from its constituent parts.
- Both analysis and synthesis are accomplished by *linear operations*, i.e., integrals and series.
- Often hear that Fourier analysis is part of "linear systems".
- *Periodic Phenomena and Fourier series*: mathematics and engineering of regularly repeating patterns.
- Periodic phenomena often are either
  - periodic in time: e.g., harmonic motion
  - periodic in space: Have phisical quantity distributed over a region with **symmetry**.
- Periodicity arises from symmetry, e.g., distribution of heat on a circular ring.
- Fourier analysis is often associated with symmetry.
- Mathematical descriptors of periodicities:
  - in time: use *frequency*, number of repititions and of patterns in one second.
  - in space: use *period*, measurement of how "big" the pattern is that *repeats*.
  - Two notions come together in, e.g., wave motion, regularly moving disturbance.
  - Again, we have frequency ν in time, cycles per second. (*fix the position* and see the variance in time.)
  - Periodicity in space, *fix the time* and see the pattern distributed over space. Length of one complete pattern is wavelength, denoted by λ.
- Relationship between frequency and wavelength: $distance = rate \times time$.
  - $v = velocity(rate)\ of\ wave$ and $\lambda = v \times \frac{1}{\nu}$ or $\lambda\nu = v$ — reciprocal relationship between frequency and wavelength.
- **The reciprocal relationship is very important!** It may help you recognize many more other relationships in two domains.
- *Math comes in because there are simple functions that are periodic and so can be used to model periodic phenomena.*
  - $\cos t$ and $\sin t$ are periodic of period 2π.
    - $\cos(t + 2\pi) = \cos t$ and $\sin(t + 2\pi) = \sin t$ — periodic in space!



- $\cos(t + 2\pi n) = \cos t$ and $\sin(t + 2\pi n) = \sin t$ for $n = 0, \pm 1, \pm 2, \ldots$ (clock-wise and counter clock-wise movement around the circle.)

Lecture 2

- *How can we use such simple functions to model complex periodic phenomena?* How general?
- Not all phenomena are periodic. Even phenomena that are periodic in time die out eventually. But periodic functions go on forever. Howeve, we can still apply ideas of periodicity. Suppose the signal looks like



even periodic phenomena, in some sense you're making an assumption there that is not really physically realizable. So even for periodic phenomena or at least functions that are periodic in time, even phenomena – soon I think I'll start talking in terms of signals rather than phenomena, but phenomena sounds a little grander at this point. Even phenomena that are periodic in time – real phenomena, they just die out eventually. We only observe – or at least we only observe something over a finite period of time, whereas, as mathematical functions, the sine and the cosine go on forever. All right. As a mathematical model, sine and cosine go on forever. So how can they really be used to model to something that dies out? But a periodic function, sine and cosine – all right – go on forever, repeating over and over again. All right. So in what sense can you really use sines and cosines to model periodic phenomena when a real periodic phenomena – when it really dies out? Well, that'll take us awhile to sort all that out. Let me just give you one answer to this, and one indication of how general these ideas really are. And you have a homework problem that asks you actually to address this mathematically. All right. So if you have a finite – you can still use – still apply ideas of periodicity, even if only as an approximation or even if only as an extra assumption. So what I mean by that is, as follows. So suppose a phenomena looks like this. Suppose the signal looks like – something like this – let me write it over here. So it dies out over a period of time. So this

- **HOW? Force periodicity by repeating the pattern.** — Periodization of a signal. See HW1.
- So the study here can be pretty general.

- Fix period for discussion and use period 1. We consider functions $f(t)$ that satisfies $f(t+1)=f(t)$ for all t. Model signals are $\cos(2\pi t)$ and $\sin(2\pi t)$.
- If we "know" a periodic function (period 1) on an interval — *any interval* of length 1. Then we know it everywhere.
- **First big idea:** One period, many frequencies.
  - e.g. $\sin(2\pi t)$ has period 1 and freq 1; $\sin(4\pi t)$ has period 1/2 and freq 2, but also period 1; $\sin(6\pi t)$ has period 1/3 and freq 3, but also period 1.
  - Combination $\sin(2\pi t)+\sin(4\pi t)+\sin(6\pi t)$ has period 1 but is composed of 3 frequencies 1,2,3.
- To model a complicated signal of period 1, we can modify amplitude, freq, and phases of $\sin(2\pi t)$ and add up results: $\sum_{k=1}^{N} A_k \sin(2\pi kt+\phi_k)$
- Different ways of wrting the sum

$$\sin(2\pi kt+\phi_k)=\sin(2\pi kt)\cos(\phi_k)+\cos(2\pi kt)\sin(\phi_k)$$
$$\sum_{k=1}^{N} A_k \sin(2\pi kt+\phi_k)=\sum_{k=1}^{N} a_k \sin(2\pi kt)+b_k\cos(2\pi kt)$$

- We can write additional constant term

$$\frac{a_0}{2}+\sum_{k=1}^{N} a_k \sin(2\pi kt)+b_k\cos(2\pi kt)$$

- By far better to represent sine and cosine via complex exponentials for the sake of convenience in math algebraically:

$$\begin{aligned} e^{2\pi ikt} &= \cos(2\pi kt)+i\sin(2\pi kt) \text{ where } i=\sqrt{-1}. \\ \text{real part: } \cos(2\pi kt) &= \frac{e^{2\pi ikt}+e^{-2\pi ikt}}{2}, \\ \text{imagine part: } \sin(2\pi kt) &= \frac{e^{2\pi ikt}-e^{-2\pi ikt}}{2i}. \end{aligned}$$

- Can convert a trig sum as before to the form:

$$\sum_{k=-N}^{N} c_k e^{2\pi ikt}$$

where c_k's are complex numbers that must satisfy symmetry property such that the summation is real: $c_{-k}=\bar{c_k}$ because

$$\begin{aligned} & c_{-k}e^{-2\pi ikt}+c_k e^{2\pi ikt} \\ =& c_{-k}e^{-2\pi ikt}+\overline{\overline{c_{-k}}e^{-2\pi ikt}} \\ =& c_{-k}e^{-2\pi ikt}+\overline{c_{-k}e^{-2\pi ikt}} \\ =& 2\mathrm{Re}(c_{-k}e^{-2\pi ikt}) \end{aligned}$$

- Can now ask fundamental question: $f(t)$ periodic of period 1. Can we write $f(t)=\sum_{k=-N}^{N} c_k e^{2\pi ikt}$?

- **Suppose** we can write $f(t) = \sum_{k=-N}^{N} c_k e^{2\pi ikt}$, **what** has to happen?
- **What** are the mystery coff's c_k in terms of f?
- Solve for the coff's.
- Isolate c_m:

$$c_m e^{2\pi imt} = f(t) - \sum_{k \neq m} c_k e^{2\pi ikt}$$
$$c_m = e^{-2\pi imt} f(t) - \sum_{k \neq m} c_k e^{2\pi i(k-m)t}$$

- Integral both sides from 0 to 1 (1 period):

$$c_m = \int_0^1 e^{-2\pi imt} f(t)dt - \sum_{k \neq m} c_k \int_0^1 e^{2\pi i(k-m)t} dt$$
$$= \int_0^1 e^{-2\pi imt} f(t)dt \qquad \text{(The second term equals zero)}.$$

- So given $f(t)$, suppose we can write $f(t) = \sum_{k=-N}^{N} c_k e^{2\pi ikt}$. Then the coff's are

$$c_m = \int_0^1 e^{-2\pi imt} f(t)dt.$$

Lecture 3

- Given $f(t)$ periodic of period 1, define k-th Fourier coeff,

$$\hat{f}(k) = \int_0^1 e^{-2\pi ikt} f(t)dt.$$

- Can we write $f(t) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi ikt}$ for some n? If so, we can analyze complex systems via simple building blocks. How general can we expect this to be?
- High stakes question. See examples below.
- EX. signal swith on for $\frac{1}{2}$ second and off for $\frac{1}{2}$ second. Can we write it via sum of building blocks? No. Not for a finite sum. Because cosine and sine is continuous and sum of them is also continuous, you cannot represent a discontinuous function by finite continuous functions.
- EX. triangle wave signal has corners. Again we cannot represent it by finite sums, because cosine and sine are differentiable and we cannot represent a non-differentiable function by finite differentiable functions.
- Any discontinuity in any derivative precludes writing

$$f(t) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi ikt}$$

because of smoothness of functions. (A **smooth function** is a function that has derivatives of all orders everywhere in its domain.)

- **Maxim** It takes high frequencies to make sharp corners.

- *To represent the general periodic signals, we have to consider infinite sums.*
- Must consider infinite sums

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$$

- Any non-smooth signal will generate infinitely many Fourier coeff's.
- Have to deal with issues of convergence.
- Need conspiracy of cancellations to make such a series converge.
- Have a summary of *main results*:
  - convergence when signal is continuous (smooth)
  - convergence when have a jump discontinuity
  - convergence in general (needs fundamental change in perspective)
- Continuous case: $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$ converges for each t to $f(t)$. (**pointwise convergence**)
- Smooth case: Again series $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$ converges to $f(t)$. (**uniform convergence**, i.e., can control the rate of convergence at different values of t depending on the degree of smoothness.)
- Jump discontinuity (e.g., switch-on-off signal): If t_0 is a point of jump discontinuity, then $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$ converges at t_0 to average of the jump, i.e., to $\frac{1}{2}(f(t_0^+) + f(t_0^-))$.
- *General case*: need a different notion of convergence. Learned to not ask for convergence of $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$ **at** value of t. *Rather*, get better picture of asking for the convergence in average sense, **convergence in the mean** or **convergence in energy**.
- Suppose $f(t)$ periodic of period 1, and suppose $\int_0^1 |f(t)|^2 dt < \infty$ (finite energy). Then you can form $\hat{f}(k) = \int_0^1 e^{-2\pi ikt} f(t)dt$. Then

$$\int_0^1 |f(t) - \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikt}|^2 dt \to 0 \text{ as } n \to \infty$$

and we write $f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$ and **notice** what the equality means (convergence in the mean).

Lecture 4 Fourier series finis

- Want to make sense of the infinite of sums $f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$
- Any lack of smoothness forces an infinite sum.
- If $f(t)$ continuous, smooth, then get a satisfactory convergence results.
- Greater generality requires a different point of view.
- Important condition is integratibility. Say $f(t)$ is square integrable $f \in L^2([0,1])$ if $\int_0^1 |f(t)|^2 dt < \infty$ (finite energy).
- If $f(t)$ is square integrable and periodic, then form $\hat{f}(k) = \int_0^1 e^{-2\pi ikt} f(t)dt$ having

$$\int_0^1 |f(t) - \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikt}|^2 dt \to 0 \text{ as } n \to \infty.$$

- Only get such convergence results if use a generalization of the integral due to Lebesgue $f \in L^2([0,1])$.
- Remember in solving for Fourier coeff's used

$$\int_0^1 e^{2\pi int} e^{-2\pi imt} dt = 0 \text{ if } m \neq n.$$

- This simple fact is corner stone for introducing "geometry" into $L^2([0,1])$. Allows one to define **orthogonality** via inner product.
- f, g square integrable on $[0,1]$.
    - Define their **inner product**, which is a generalization of dot product for vectors

$$(f,g) = \int_0^1 f(t)\overline{g(t)}dt$$

    which is the continuous, infinite dimension of dot product.

- f and g are **orthogonal** if $(f,g) = 0$.
- **Norm** of f is $(f,f) = ||f||^2 = \int_0^1 |f(t)|^2 dt$.
- *Pythogonal theorem*: f is orthogonal to g
    - if and only if $||f+g||^2 = ||f||^2 + ||g||^2$
    - if and only if $(f,g) = 0$.
    - This comes from vectorization.
- $(e^{2\pi int}, e^{2\pi imt}) = 0 \text{ if } m \neq n \text{ and } 1 \text{ otherwise.}$.
- Use inner product to define and compute projection. Given u, v unit vector, the projection vector is $(u,v)u$.
- For functions, it is unable to draw any picture to see the orthogonality.
- Fourier coeff's is exactly the projection

$$\begin{aligned}(f(t), e^{2\pi ikt}) &= \int_0^1 f(t)\overline{e^{2\pi ikt}}dt \\ &= \int_0^1 f(t)e^{-2\pi ikt}dt \\ &= \hat{f}(k)\end{aligned}$$

- To write $f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt}$ is to write

$$f(t) = \sum_{k=-\infty}^{\infty} (f(t), e^{2\pi ikt})e^{2\pi ikt}$$

- Express f in terms of its components.
- To write this is to say that the complex exponentials $\{e^{2\pi ikt}\}$ for $-\infty < k < \infty$ form an orthogonal basis for $L^2([0,1])$.*
- *Rayleigh's Identity*

$$\int_0^1 |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$$

which says the length of a vector is the sum of the square of its components.

- The energy of the complex exponential is 1, so the total energy depends on the Fourier coeff's.
- *Application to heat flow*
  - Have a region in space and an initial distribution $f(t)$ of temperature.
  - How does the temperature change both in position and time?
  - Periodicity comes from the periodicity of space.
  - Look at a heated ring with initial temperature $f(x)$ where x is the point on the circle. $u(x,t)$ is temperature at x at time t.
  - Want to study $u(x,t)$.
  - Temperature is periodic as a function of x. Suppose period 1,

    $f(x+1) = f(t)$ and $u(x+1,t) = u(x,t)$.
  - We write Fourier series:

$$u(x,t) = \sum_{k=-\infty}^{\infty} c_k(t) e^{2\pi ikx}$$

    Time dependence is in c_k.
  - What are the c_k?
  - Have *heat equation* $u_t = au_{xx}$ n-dim heat equation. (Diffusion equation) Use this for the ring.
  - Choose $a = \frac{1}{2}$ so $u_t = \frac{1}{2}u_{xx}$ and plug $u(x,t)$ into equation

$$\begin{aligned} u_t &= \sum_k c_k'(t) e^{2\pi ikx} \\ u_{xx} &= \sum_k c_k(t)(2\pi ik)^2 e^{2\pi ikx} \\ &= \sum_k c_k(t)(-4\pi^2 k^2) e^{2\pi ikx} \\ \sum_k c_k'(t) e^{2\pi ikx} &= \sum_k c_k(t)(-2\pi^2 k^2) e^{2\pi ikx} \\ c_k'(t) &= c_k(t)(-2\pi^2 k^2) \\ c_k(t) &= c_k(0) e^{-2\pi^2 k^2 t} \end{aligned}$$

  - At $t = 0$,

$$f(x) = u(x,0) \\ = \sum_k c_k(0)e^{2\pi ikx}$$

This must be the Fourier series of f(x).

- $u(x,t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{-2\pi^2k^2t}e^{2\pi ikx}$
- As $t \to 0$, $u(x,t) \to 0$. The ring finally gets cool.

Lecture 5

- $u(x,t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{-2\pi^2k^2t}e^{2\pi ikx}$
- Write this differently. Use

$$\hat{f}(k) = \int_0^1 e^{-2\pi iky}f(y)dy$$

$$u(x,t) = \int_0^1 (\sum_{k=-\infty}^{\infty} e^{-2\pi iky}e^{-2\pi^2k^2t}e^{2\pi ikx})f(y)dy \\ = \int_0^1 (\sum_{k=-\infty}^{\infty} e^{2\pi ik(x-y)}e^{-2\pi^2k^2t})f(y)dy \\ = \int_0^1 g(x-y,t)f(y)dy$$

where $g(x,t) = \sum_{k=-\infty}^{\infty} e^{2\pi ikx}e^{-2\pi^2k^2t}$.

This expresses $u(x,t)$ as **convolution** of $f(x)$ with heat kernel $g(x,t)$.

- $g(x,t)$ is called *heat kernel*, fundamental solution of heat equation, and Green's function for heat equation.
- For any problem that has to do with Fourier analysis, convolution comes into play.

Transition from Fourier series to Fourier transforms.

- This is the transition from periodic phenomena to non-periodic phenomena.
- Will do this by viewing non-periodic function as a limiting case of periodic function as period tends to infinity. It takes a little work.
- Two aspects
  - (analysis) $f(t)$ periodic, $\hat{f}(k) = \int_0^1 e^{-2\pi ikt}f(t)dt$: decomposes the function into the constituent parts.
  - (synthesis) $f(t) = \sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikt}$.
- **Fourier transform** is generalisation (limiting case) of Fourier coeff's (analysis). **Inverse Fourier transform** is the generalization of Fourier series (synthesis).
- Need set up when $f(t)$ periodic of period T. (Ultimately want T tends to ∞.)
- **Building blocks** :

$$e^{2\pi it(k/T)}$$

Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{2\pi i(k/T)t}$$

- What are the coeff's?

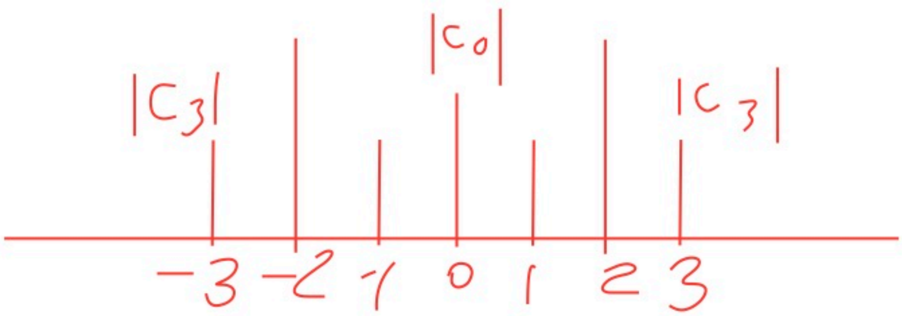
$$c_k = \frac{1}{T}\int_0^T e^{-2\pi i(k/T)t} f(t)dt$$

  Write this

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2\pi i(k/T)t} f(t)dt$$

- Picture of frequencies?

$$c_{-k} = \overline{c_k}$$
$$|c_{-k}| = |\overline{c_k}| = |c_k|$$



  - Spacing of frequency is 1.
  - What about period T? Spacing of frequency is $\frac{1}{T}$.
  - Reciprocal relationship between period (time domain) and frequencies (frequency domain). Period T Frequency $\frac{1}{T}$ apart.
  - $T < 1$: spacing bigger and spectrum stretched out
  - $T > 1$: spacing smaller and spectrum squeezed
  - $T \to \infty$: spectrum becomes "*continuous*", $\frac{1}{T} \to 0$.
  - **However, $T \to \infty$ leads Fourier coeff's to be zero.**

Lecture 6

- Fourier transform is as a limiting case of Fourier coeff's and Fourier series.
- Periodize to make periodic of period T (think of T as big). But it does not work because $c_k \to 0$ as $T \to \infty$.
- Scale up by T.
  - Write $Ff(k/T) = \int_{-T/2}^{T/2} e^{-2\pi i(k/T)t} f(t)dt$
  - Fouries series looks like

$$f(t)=\sum_{k=-\infty}^{\infty} Ff(\frac{k}{T})e^{2\pi i(k/T)t}\frac{1}{T}$$

- Now let $T\to\infty$, "discrete variables" replaced by "continuous variable" s, $-\infty<s<\infty$

$$Ff(s)=\int_{-\infty}^{\infty} e^{-2\pi ist}f(t)dt$$
$$f(t)=\int_{-\infty}^{\infty} Ff(s)e^{2\pi ist}ds$$

- If f(t) a function defined for $-\infty<t<\infty$, define its **Fourier transform** as

$$Ff(s)=\int_{-\infty}^{\infty} e^{-2\pi ist}f(t)dt$$

- Will need to understand convergence of the integral to well define Fourier transform.
- Fourier transform analyzes f(t) into its constituent parts.
- Fourier inversion says that we can synthesize f(t) from its constituent parts,

$$f(t)=\int_{-\infty}^{\infty} Ff(s)e^{2\pi ist}ds$$

- f(t) time domain, Ff(s) frequency domain.
- **Major Secret of the Universe**: Every signal has a spectrum. And the spectrum determines the signal.

$$Ff(s)=\int_{-\infty}^{\infty} e^{-2\pi ist}f(t)dt$$
$$F^{-1}g(t)=\int_{-\infty}^{\infty} e^{2\pi ist}g(s)ds$$
$$F^{-1}Ff=f$$
$$FF^{-1}g=g$$
$$Ff(0)=\int_{-\infty}^{\infty} f(t)dt$$
$$F^{-1}g(0)=\int_{-\infty}^{\infty} g(s)ds$$

- **Examples** (straight integral)
  - Rectangle function (Indication function) $F\pi(s)=sinc(s)=\frac{\sin\pi s}{\pi s}$.
  - Triangle function $Ff(s)=sinc^2(s)$.

Lecture 7

- Question on existence (convergence) of integral. — later
- Set of $S\in\mathcal{R}$ for which $Ff(s)$ is defined is called the **spectrum**.
- Definitions and notations vary. You have to distinguish which is the convention under certain context.

- Examples:
  - Gaussian

$$f(t) = e^{-\pi t^2}$$
$$\int_{-\infty}^{\infty} e^{-\pi t^2}\, dt = 1$$

  - *Fourier transform of Gaussian is itself!*

$$\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} e^{-2\pi i s t} e^{-\pi t^2}\, dt \\
F'(s) &= \int_{-\infty}^{\infty} \frac{d}{ds} e^{-2\pi i s t} e^{-\pi t^2}\, dt \\
&= \int_{-\infty}^{\infty} -2\pi i t e^{-2\pi i s t} e^{-\pi t^2}\, dt \\
&= i \int_{-\infty}^{\infty} e^{-2\pi i s t} (-2\pi t e^{-\pi t^2}) dt \\
&= i\{[e^{-2\pi i s t} e^{-\pi t^2}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-2\pi i s t} (-2\pi i s e^{-\pi t^2}) dt\} \\
&= -\int_{-\infty}^{\infty} 2\pi s e^{-2\pi i s t} e^{-\pi t^2}\, dt \\
&= -2\pi s F(s) \\
F(s) &= F(0) e^{-\pi s^2} \\
F(0) &= 1 \\
F(s) &= e^{-\pi s^2}
\end{aligned}$$

- **Fourier transform duality —** exploit the similarity between the formulas for Fourier transform and inverse.

$$\begin{aligned}
Fs(-s) &= \inf_{-\infty}^{\infty} e^{2\pi i s t} f(t) dt \\
&= F^{-1} f(s) \\
Fs(-s) &= F^{-1} f(s) \quad \text{(Operational notation)}
\end{aligned}$$

- Write this more neatly.
  - Introduce reverse signal; Define $f^{-1}(t) = f(-t)$.
  - If f is even, $f^{-1} = f$; If f is odd, $f^{-1} = -f$.

$$\begin{gathered}
Ff(-s) = F^{-1} f(s) \\
(Ff)^{-1}(s) = Ff(-s) = F^{-1} f(s) \\
(Ff)^{-1} = F^{-1} f \\
F(f^{-1}) = F^{-1} f \\
(Ff)^{-1} = F^{-1} f = F(f^{-1}) \\
FFf = f^{-1}
\end{gathered}$$

- Application
  - Find $Fsinc()$

$$F\pi = sinc$$
$$FF\pi = Fsinc = \pi^{-1} = \pi$$

- Same duality argument gives $Fsinc^2 = tri$.

Lecture 8 — Delays, sketches, and convolutions

- If a signal is *delayed* (shifted) by an amount of b, what happens to Fourier transform?

$$f(t) \leftrightarrow F(s)$$
$$f(t-b) \leftrightarrow e^{-2\pi isb}F(s)$$
$$f(t \pm b) \leftrightarrow e^{\pm 2\pi isb}F(s)$$

- Fourier transform is a complex number, so it has its *magnitude* and *phase*. Write

$$F(s) = |F(s)|e^{2\pi i\theta(s)}$$
$$e^{-2\pi isb}F(s) = |F(s)|e^{2\pi i(\theta(s)-sb)}$$

- *Scaling*

$$f(t) \leftrightarrow F(s)$$
$$f(at) \leftrightarrow \frac{1}{a}F(\frac{s}{a}) \quad \text{if } a > 0$$
$$f(at) \leftrightarrow -\frac{1}{a}F(\frac{s}{a}) \quad \text{if } a < 0$$
$$f(at) \leftrightarrow \frac{1}{|a|}F(\frac{s}{a})$$

- *Interpretation*
  - If a>1, f(at) squeezed and F(s) stretched out horizontally and squashed vertically. (Spectrum only displays the magnitude.)
  - If a<1, f(at) stretched out and F(s) squeezed horizontally and stretched vertically.
  - **You cannot localize the signal both in time and frequency at the same time.**
- Recall situation with Gaussian $f(t) = e^{-\pi t^2}$, $F(s) = e^{-\pi s^2}$. Perfectly balanced in time and frequency.
- **Convolution**—probably the most frequent operation in signal processing.
- Basic question in all signal processing — How to use one function to modify another.
  - Most often look to modifying the spectrum of a signal.
  - E.g. Linearity

$$F(f+g) = Ff + Fg$$

  - Modify Ff by adding Fg.
  - What about multiplying?

$$
\begin{aligned}
(Ff)(Fg) &= F(f \text{ combined with } g)? \\
(Ff)(Fg) &= (\int_{-\infty}^{\infty} e^{-2\pi isx} f(x)dx)(\int_{-\infty}^{\infty} e^{-2\pi isy} g(y)dy) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi is(x+y)} f(x)g(y)dxdy \\
&= \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} e^{-2\pi isu} f(u-y)du)g(y)dy \\
&= \int_{-\infty}^{\infty} e^{-2\pi isu}(\int_{-\infty}^{\infty} f(u-y)g(y)dy)du \\
&= F\int_{-\infty}^{\infty} f(u-y)g(y)dy
\end{aligned}
$$

- Define **convolution** of f and g by

  $$(g * f)(x) = \int_{-\infty}^{\infty} g(x-y)f(y)dy$$

  Then we have

  $$F(f * g) = (Ff)(Fg).$$

Lecture 9

- Led to the convolution by asking what the combination of f and g results in its Fourier transform as multiplication.
- Example of this in filtering.
  - Kill off high frequency by multiply in freq domain by a scaled rectangle function. (low pass filter)
  - In time domain, this is convolution.
  - Filtering is often synonomous with convolution.
    - Filter is a system that involves an input with a **fixed** function called the impulse response.

      $$
      \begin{aligned}
      g &= f * h \\
      G(s) &= F(s)H(s)
      \end{aligned}
      $$

      where $H(s)$ called *transfer function*.
    - To design a filter is to design $H(s)$.
    - *Easy to understand filtering (convolution) in frequency; not so easy in time.*
- Need to visualize convolution.
  - Don't even try! Hard to flip and drag.
  - If we can't visualize convolution easily. Is there a good interpretation?
  - *SUGGESTION*: Used in many ways. Not subject to single interpretation.
- **Maxiom**: *In many contexts, convolution is associated with smoothing or averaging.*

- **Axiom**: In general, $f * g$ has the best properties of f and g separately. $f * g$ is "smoother" than f and g separately.
  - Ex. $\pi * \pi = tri$ discontinuous (right) continuous (left)
  - f differentiable, g not. (f*g) is differentiable and (f*g)'=f'*g. Similarly with higher derivatives.
- *Convolution and Differential Equations*:
  - **Thm.** the derivative for Fourier transform

$$F(f')(s) = 2\pi i s F f(s)$$

    Fourier transform turns differential operation into product operation.

  - Similarly,

$$F(f^n)(s) = (2\pi i s)^n F f(s)$$

  - Derivation in case when $f(t) \to 0$ as $t \to \pm\infty$.

$$\begin{aligned} F(f')(s) &= \int_{-\infty}^{\infty} e^{-2\pi i s t} f'(t) dt \\ &= e^{-2\pi i s t} f(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-2\pi i s) e^{-2\pi i s t} f(t) dt \\ &= 0 + (2\pi i s) \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t) dt \\ &= 2\pi i s F f(s) \end{aligned}$$

  - Heat equation on infinite rod.

$$\begin{aligned} \text{Heat equation} : u_t &= \frac{1}{2} u_{xx} \\ u(x,0) &= f(x) \\ \text{Take Fourier transform in spatial variable} \\ u(x,t) &\to U(s,t) \\ F u_t &= \int e^{-2\pi i s x} \frac{d}{dt} u(x,t) dx \\ &= \frac{d}{dt} U(s,t) \\ F(u''(x,t))(s) &= -4\pi^2 s^2 U(s,t) \\ \frac{d}{dt} U(s,t) &= -2\pi^2 s^2 U(s,t) \\ U(s,t) &= U(s,0) e^{-2\pi^2 s^2 t} \\ U(s,t) &= F(s) e^{-2\ pi^2 s^2 t} \\ F(\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}) &= e^{-2\pi^2 s^2 t} \\ u(x,t) &= f(x) * (\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}) \end{aligned}$$

Lecture 10 Convolution and Central Limit Theorem

- CLT explains universal appearance of bell shaped curve (Gaussian) in probability.

- Most probabilities: some kind of average are calculated (approximated) as if they are determined by a Gaussian.

- We've used $e^{-\pi t^2}$ as "standard" Gaussian. For CLT the normalization is to take

$$P(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

  - It has mean zero and standard deviation 1.

$$Prob(a \leq measurement \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

- *Set-up*
  - "Primitive notion" is *random variable* and how it is distributed.
  - Call random variable X and measurement (value of random variable) x.
  - Interested in how measurements X are distributed given by a function $p(x) \geq 0$ ($P_X(x)$)

$$Prob(a \leq X \leq b) = \int_a^b p(x)dx$$
$$Prob(-\infty < X < \infty) = 1$$

  - **Key results**: X_1 and X_2 are independent and distribution $p_1(x_1)$ and $p_2(x_2)$. The distribution of $X_1 + X_2$ is given by $p_1 * p_2$

$$\begin{aligned} Prob(X_1 + X_2 \leq t) &= \iint_{x_1+x_2 \leq t} p_1(x_1)p_2(x_2)dx_1dx_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{t} p_1(u)p_2(v-u)dvdu \\ &= \int_{-\infty}^{t} (\int_{-\infty}^{\infty} p_1(u)p_2(v-u)du)dv \\ &= \int_{-\infty}^{t} p_1 * p_2(v)dv \end{aligned}$$

    This identifies $p_1 * p_2$ as distribution of $X_1 + X_2$!!
  - Similar result for $X_1 + X_2 + \ldots + X_n$ distribution to $p_1 * p_2 * \ldots * p_n$.
  - n independent random variables. Also assume that distributions are all the same for all X_i.
  - Say $X_1, \ldots, X_n$ are iid (independent, identically distributed). Call the distribution $p(x)$. Can normalize further. Mean 0 and std deviation is 1.
  - What is the distribution of $S_n = X_1 + \ldots + X_n$? Mean 0 and std dev of S_n is $\sqrt{n}$.
  - CLT:

$$\lim_{n \to \infty} Prob(a \leq \frac{S_n}{\sqrt{n}} \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

  - Will show an unintegrated form of this.

$$p_n(x) \sim p(\frac{X_1 + \ldots + X_n}{\sqrt{n}})$$

$$\text{Then } p_n(x) \to \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ as } n \to \infty$$

$$p_n(x) = \sqrt{n} p^{*n}(\sqrt{n} x)$$

$$\begin{aligned}
F(p_n(x)) &= \sqrt{n} \times \frac{1}{\sqrt{n}} F^n(\frac{s}{\sqrt{n}}) \\
&= F^n(\frac{s}{\sqrt{n}}) \\
F(\frac{s}{\sqrt{n}}) &= \int_{-\infty}^{\infty} e^{-2\pi i \frac{s}{\sqrt{n}} x} p(x) dx \\
&\approx \int_{-\infty}^{\infty} (1 - \frac{2\pi i s x}{\sqrt{n}} - \frac{2\pi^2 s^2 x^2}{n}) p(x) dx \\
&= \int_{-\infty}^{\infty} p(x) dx - \frac{2\pi i s}{\sqrt{n}} \int_{-\infty}^{\infty} x p(x) dx - \int_{-\infty}^{\infty} \frac{2\pi^2 s^2 x^2}{n} p(x) dx \\
&= 1 - \frac{2\pi^2 s^2}{n} \\
F^n(\frac{s}{\sqrt{n}}) &= (1 - \frac{2\pi^2 s^2}{n})^n \\
&\approx e^{-2\pi^2 s^2}
\end{aligned}$$

The inverse Fourier transform gives CLT!